MORSE HOMOLOGY FOR GENERATING FUNCTIONS OF LAGRANGIAN SUBMANIFOLDS

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ABSTRACT. The purpose of the paper is to give an alternative construction and the proof of the main properties of symplectic invariants developed by Viterbo. Our approach is based on Morse homology theory. This is a step towards relating the "finite dimensional" symplectic invariants constructed via generating functions to the "infinite dimensional" ones constructed via Floer theory in Y.-G. Oh, *Symplectic topology as the geometry of action functional.* I, J. Diff. Geom. **46** (1997), 499–577.

1. INTRODUCTION

Let M be a compact smooth manifold and $\pi_{T^*M} : T^*M \to M$ its cotangent bundle. Consider the space of paths in T^*M starting at the zero section

$$\Omega := \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M\}$$

as a fibration over M, given by

$$\pi_\Omega : \Omega \to M, \quad \pi_\Omega(\gamma) := \pi_{T^*M}(\gamma(1)).$$

For a smooth function (Hamiltonian) $H : T^*M \times [0,1] \to \mathbb{R}$ the classical action functional

$$\mathcal{A}_H : \Omega \to \mathbb{R}$$

is defined by

$$\mathcal{A}_H = \int_\gamma \theta - \int_0^1 H(\gamma(t), t) dt, \tag{1}$$

where $\theta := \sum p_i dq_i$ is the canonical one-form on T^*M.

If ξ is a vector field along $\gamma \in \Omega$, the first variation of $\mathcal{A}_H$ in ξ-direction is

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 \left[\omega(\frac{d\gamma}{dt}, \xi) - dH(\gamma(t), t)\xi\right] dt + \theta(\xi(1)),$$

where $\omega := -d\theta$ is the canonical symplectic form on T^*M. The fiber derivative of $\mathcal{A}_H$ vanishes on the set

$$\Sigma_{\mathcal{A}_H} := \{\gamma \in \Omega \mid \frac{d\gamma}{dt} \lrcorner \omega = dH(\gamma)\},$$

i.e. on the set of Hamiltonian orbits in T^*M. If ϕ_t^H is the Hamiltonian isotopy generated by H, then it is easy to see that

$$\phi_1^H(o_M) = \{d\mathcal{A}_H(\gamma) \mid \gamma \in \Sigma_{\mathcal{A}_H}\}.$$

The action functional (1) can be considered as a special case of the following more general construction. Let $\pi : E \to M$ be a smooth fibration over M with fibers $F_e := \pi^{-1}(\pi(e))$. Let $S : E \to \mathbb{R}$ be a smooth function such that $dS(e) \pitchfork \nu^* F_e$ for any $e \in E$, where $\nu^* F_e$ is the conormal bundle of F_e in T^*E. Then

$$\Sigma_S := \{e \in E \mid dS(e) \in \nu^* F_e\}$$

is a smooth submanifold of E of dimension $n = \dim M$. We define a map

$$i_S : \Sigma_S \to T^*M$$

by

$$i_S(e) = (T^*\pi)^{-1}(dS(e)).$$

It is easy to prove that the mapping $i_S : \Sigma_S \to T^*M$ is an exact Lagrangian immersion. Moreover,

$$i_S^*\theta = d(S|_{\Sigma_S}).$$

Definition 1. If $L = i_S(\Sigma_S)$, we call S the generating function of an (immersed) Lagrangian submanifold $L \subset T^*M$. If E is a vector bundle and $S - Q$ has compact support for some non-degenerate fiberwise quadratic form on E, then S is called the generating function quadratic at infinity.

The idea of representing Lagrangian submanifolds through generating functions on a finite dimensional space was used by Hörmander in [9] in calculus of Fourier integral operators. Chaperon [2, 3] and Laudenbach and Sikorav [10] in the proof of the Arnold conjecture for Lagrangian intersections in the cotangent bundle used the method of "broken trajectories" of Hamiltonian paths to construct the generating functions for certain Lagrangian submanifolds in T^*M as a finite dimensional version of the action functional (1).

Theorem 2 ([10], [16]). *If $L \subset T^*M$ is an embedded Lagrangian submanifold having a generating function quadratic at infinity and $L_t := \phi_t^H(L)$ for some Hamiltonian isotopy ϕ_t^H, then there exist a vector bundle E_1 and a family $S_t : E \to \mathbb{R}$ of functions such that S_t generates L_t for all t. In particular, Hamiltonian deformation of zero section admits a generating function quadratic at infinity.*

The theory of generating functions was further developed by Sikorav [16], Theret [18], Traynor [19], Viterbo [20] and many other authors. If $S : E \to \mathbb{R}$ is a generating function for $L \subset T^*M$, then the following modifications of S also generate L:

1. *Stabilization $\tilde{S} : E \oplus F \to \mathbb{R}$ of S:*

$$\tilde{S}(q, \zeta, \eta) = S(q, \zeta) + Q(\eta)$$

   for some non-degenerate fiberwise quadratic form Q on F.
2. *Gauge equivalence:*

$$\tilde{S} = S \circ \Phi$$

   for some fiber-preserving diffeomorphism $\Phi : E \to E$.

3. *Addition of a constant*:

$$\tilde{S} = S + c_0$$

for some constant $c_0 \in \mathbb{R}$.

The choice of a generating function for a given Lagrangian submanifold $L \subset T^*M$ is unique up to these three transformations:

Theorem 3 (Viterbo, [20]). *Let S_1 and S_2 be two generating functions quadratic at infinity generating the same embedded Lagrangian submanifold $L = \phi_1^H(o_M)$ in T^*M. Then, after stabilization, S_1 is gauge equivalent to $S_2 + c_0$ for some $c_0 \in \mathbb{R}$.*

Note that the after modifications 1.–3. a new S does not have to be quadratic at infinity. However, after a gauge transformation which leaves S unchanged on a prescribed compact set, we can obtain a generating function quadratic at infinity. More precisely, we have the following

Proposition 4 (Proposition 11 [18]). *If Q is a fiberwise quadratic form on E and $S : E \to \mathbb{R}$ is a function such that $\|\frac{\partial}{\partial q}(S - Q)\|$ is bounded, then for every compact set $K \subset E$ there exists a fiber preserving diffeomorphism $\Phi : E \to E$ such that $\Phi = id$ on K and $S \circ \Phi = Q$ at infinity.* □

In [20] Viterbo developed a rather sophisticated geometric and topological theory of generating functions by studying deformations of their level sets. For set $\mathcal{S}$ of generating functions quadratic at infinity he defined the function

$$c : H^*(M) \times \mathcal{S} \to \mathbb{R}$$

and proved that it is essentially independent of the choice of generating function in the class of *normalized* functions generating a fixed Lagrangian submanifold. Furthermore, he applied this theory to

$$L := \mathrm{Graph}(\phi) \subset \mathbb{C}^n \times \mathbb{C}^n$$

to construct a biinvariant norm on the Lie group of compactly supported Hamiltonian diffeomorphisms in $\mathbb{C}^n$. That construction has a number of interesting implications to the symplectic geometry of $\mathbb{C}^n$: the construction of a symplectic capacity, the construction of a certain biinvariant norm on the group of Hamiltonian diffeomorphisms, a proof of the "Camel Theorem", the existence of infinitely many periodic points of compactly supported symplectic diffeomorphism in the interior of its support and others.

The infinite dimensional analogue of Viterbo's invariants was developed by Oh [14] by studying the deformation of level sets of the action functional by means of Floer theory. That construction has some applications to Hofer's geometry.

In this paper we will give a construction alternative to Viterbo's, from the point of view of Morse homology. This is a step towards relating the "finite dimensional" symplectic invariants constructed via generating functions to the "infinite dimensional" ones constructed via Floer theory (see [11]).

2. MORSE HOMOLOGY

We briefly recall the main points of the construction of Morse homology and refer the reader to [15] for a detailed description. The idea of this construction goes back to Smale [17] and Milnor [12] and has been reformulated by Witten [21] and

Floer [4]–[6]. For a Morse function f on a compact smooth manifold M we denote by $\mathrm{Crit}_p(f)$ the set of its critical points of index p and define

$$C_p(f) := \text{ free abelian group generated by } \mathrm{Crit}_p(f).$$

Consider the gradient flow generated by the equation

$$\frac{d\gamma}{dt} = -\nabla f, \tag{2}$$

where ∇f is defined with respect to the given Riemannian metric g on M. Denote by $\mathcal{M}_{f,g}(M)$ the set of all $\gamma : \mathbb{R} \to M$ satisfying (2) such that

$$\int_{-\infty}^{+\infty} \left| \frac{d\gamma}{dt} \right|^2 dt < \infty.$$

For a generic choice of a metric, the spaces

$$\mathcal{M}_{f,g}(x^-, x^+) = \{\gamma \in \mathcal{M}_{f,g}(M) \mid \gamma(t) \to x^{\pm} \text{ as } t \to \pm\infty\}$$

are smooth manifolds of dimension $m(x^+) - m(x^-)$, where $m(x)$ denotes the Morse index of a critical point x. Note that

$$\mathcal{M}_{f,g}(x, y) \cong W_g^u(x) \cap W_g^s(y),$$

where $W_g^s(y)$ and $W_g^u(x)$ are the stable and unstable manifolds of the gradient flow (2). For generic g the intersection above is transverse (Morse-Smale condition). The group $\mathbb{R}$ acts on $\mathcal{M}_{f,g}(x, y)$ by $\gamma \mapsto \gamma(\cdot + t)$. We denote

$$\widehat{\mathcal{M}}_{f,g}(x, y) := \mathcal{M}_{f,g}(x, y)/\mathbb{R}.$$

The manifolds $\widehat{\mathcal{M}}_{f,g}(x, y)$ can be given a coherent (i.e. compatible with gluing process) orientation σ (see [15]).

We define

$$\partial : C_p(f) \to C_{p-1}(f),$$

$$\partial x := \sum_{y \in \mathrm{Crit}_{p-1}(f)} n(x, y)y,$$

where $n(x, y)$ is the number of points in zero dimensional manifold $\widehat{\mathcal{M}}_{f,g}(x, y)$ counted with the sign with respect to the orientation σ. The proof of $\partial \circ \partial = 0$ is based on gluing and cobordism arguments (see, for example, [15]). Now Morse homology groups are defined by

$$H_p^{\mathrm{Morse}}(f) := \mathrm{Ker}(\partial)/\mathrm{Im}(\partial).$$

For generic choices of Morse functions f_1 and f_2 the groups $H_p(f_1)$ and $H_p(f_2)$ are isomorphic. Furthermore, they are isomorphic to the singular homology group of M, i.e.

$$H_p^{\mathrm{Morse}}(f) \cong H_p^{\mathrm{sing}}(M)$$

for generic f. We refer the reader to [12] for the proof.

Wherever there is no possibility of confusion we will omit the superscript *Morse* in the above notation. Also, we will use the symbol $H_p(M)$ to denote the *Morse homology* defined by means of some Morse function $f : M \to \mathbb{R}$, wherever it is not necessary to specify f. Furthermore, when it is necessary to emphasize the role of a particular Riemannian metric g with respect to which the gradient in 2 is defined, we denote the Morse homology groups by $H_p(g, f)$. Finally, when it is necessary to

emphasize the coherent orientation σ used in a definition of a boundary operator ∂, we will write $H_p^\sigma(f)$.

For a later purpose we briefly sketch the construction of isomorphism

$$(3) \qquad h_{\alpha\beta} : H_p(f^\alpha) \to H_p(f^\beta)$$

for generic Morse functions f^α, f^β. Consider the "connecting trajectories", i.e. the solutions of non-autonomous equation

$$(4) \qquad \frac{d\gamma}{dt} = -\nabla f_t^{\alpha\beta}$$

where $f_t^{\alpha\beta}$ is a homotopy connecting f^α and f^β such that for some $R > 0$

$$f_t^{\alpha\beta} \equiv f^\alpha \text{ for } t \leq -R,$$

$$f_t^{\alpha\beta} \equiv f^\beta \text{ for } t \geq R.$$

For $x^\alpha \in \mathrm{Crit}_p(f^\alpha)$ and $x^\beta \in \mathrm{Crit}_p(f^\beta)$ denote

$$\mathcal{M}_{f^{\alpha\beta},g}(x^\alpha, x^\beta) := \{\gamma \mid \gamma \text{ satisfies (4) and } \lim_{t\to-\infty} \gamma = x^\alpha, \ \lim_{t\to\infty} \gamma = x^\beta\}.$$

As before, $\mathcal{M}_{f^{\alpha\beta},g}$ is a smooth finite dimensional manifold. Now, define

$$(h_{\alpha\beta})_\sharp : C_p(f^\alpha) \to C_p(f^\beta)$$

by

$$(h_{\alpha\beta})_\sharp x^\alpha = \sum_{x^\beta \in \mathrm{Crit}_p(f^\beta)} n(x^\alpha, x^\beta) x^\beta, \text{ for } x^\alpha \in \mathrm{Crit}_p(f^\alpha),$$

where $n(x^\alpha, x^\beta)$ is the algebraic number of points in zero dimensional manifold $\mathcal{M}_{f^{\alpha\beta},g}(x^\alpha, x^\beta)$ counted with the signs defined by the orientation of $\mathcal{M}_{f^{\alpha\beta},g}$. Homomorphisms $(h_{\alpha\beta})_\sharp$ commute with ∂ and thus define the homomorphisms $h_{\alpha\beta}$ in homology which, in addition, satisfy $h_{\alpha\beta} \circ h_{\beta\gamma} = h_{\alpha\gamma}$. Proof of these facts is similar to the proof of $\partial^2 = 0$ (see [15] for more details).

In the previous construction we fixed the Riemannian metric g on M. If we fix a Morse function $f : M \to \mathbb{R}$ instead, we establish the isomorphism

$$h_{\alpha\beta} : H_p(g^\alpha, f) \to H_p(g^\beta, f)$$

between the two Morse homology groups defined by means of two generic metrics g^α and g^β in a similar way, by considering the "connecting trajectories",

$$(5) \qquad \frac{d\gamma}{dt} = -\nabla^{g_t^{\alpha\beta}} f.$$

Here $g_t^{\alpha\beta}$ is a homotopy connecting g^α and g^β such that for some $R > 0$

$$g_t^{\alpha\beta} \equiv g^\alpha \text{ for } t \leq -R,$$

$$g_t^{\alpha\beta} \equiv g^\beta \text{ for } t \geq R,$$

and ∇^g is a gradient defined by metric g.

Note that f is decreasing along the trajectories solving autonomous gradient equation (2). Therefore, the boundary operator ∂ preserves the downward filtration given by level sets of f. In other words, if we denote

$$\mathrm{Crit}_p^\lambda(f) := \mathrm{Crit}_p(f) \cap f^{-1}((-\infty, \lambda])$$

and
$$C_p^\lambda(f) := \text{ free abelian group generated by } \text{Crit}_p^\lambda(f),$$
then the boundary operator ∂ restricts to
$$\partial^\lambda : C_p^\lambda(f) \to C_{p-1}^\lambda(f).$$
Obviously, $\partial^\lambda \circ \partial^\lambda = 0$ and hence we can define the relative Morse homology groups
$$H_p^\lambda(f) := \text{Ker}(\partial^\lambda)/\text{Im}(\partial^\lambda).$$
An obvious inclusion
$$j^\lambda : \text{Crit}_p^\lambda(f) \to \text{Crit}_p(f)$$
generates the homomorphism
$$j_\sharp^\lambda : C_p^\lambda(f) \to C_p(f)$$
which commutes with ∂, i.e. $\partial^\lambda \circ j_\sharp^\lambda = j_\sharp^\lambda \circ \partial$. Hence, we have an inclusion homomorphism
$$j_*^\lambda : H_p^\lambda(f) \to H_p(f).$$

Following the standard algebraic construction, we define (relative) Morse cohomology. We set
$$C_\lambda^p(f) := \text{Hom}(C_p^\lambda(f), \mathbb{Z})$$
and
$$\delta^\lambda : C_\lambda^p(f) \to C_\lambda^{p+1}(f), \quad \langle \delta^\lambda a, x \rangle := \langle a, \partial^\lambda x \rangle$$
and define
$$H_\lambda^p(f) := \text{Ker}(\delta^\lambda)/\text{Im}(\delta^\lambda).$$
Since $\text{Crit}_p(f)$ is finite, for large λ we have $H_p^\lambda(f) = H_p(f)$ and $H_\lambda^p(f) = H^p(f)$. Homomorphism $j_\sharp^\lambda$ yields
$$j_\lambda^\sharp C^p(f) \to C_\lambda^p(f), \quad \langle j_\lambda^\sharp a, x \rangle := \langle a, j_\sharp^\lambda x \rangle.$$
Hence, we have the restriction homomorphism
$$j_\lambda^* : H^p(f) \to H_\lambda^p(f).$$

Definition of Morse homology and cohomology with coefficients in arbitrary ring R is straightforward.

3. POINCARÉ DUALITY AND COHOMOLOGY OPERATIONS

3.1. **Poincaré duality.** Let R be a ring and let M be a compact R-oriented smooth manifold of dimension n. We will assume that all homology and cohomology groups in this section are with coefficients in R. Let $f : M \to \mathbb{R}$ be a Morse function. The obvious bijection
$$\text{Crit}_p(-f) \to \text{Crit}_{n-p}(f), \ x \mapsto x$$
gives rise to an isomorphism
$$\Gamma : C_p(-f) \to C^{n-p}(f),$$
$$\Gamma x = \delta_x, \text{ for } x \in \text{Crit}_p(-f).$$

The correspondence

$$\Psi : \gamma \mapsto \tilde{\gamma}, \ \ \tilde{\gamma}(t) := \gamma(-t)$$

gives rise to the diffeomorphism

$$\Psi : \mathcal{M}_{f,g}(y,x) \overset{\cong}{\to} \mathcal{M}_{-f,g}(x,y).$$

Let σ be a canonical coherent orientation of $\mathcal{M}_{f,g}(M)$ and let $\tilde{\sigma}$ denote a push-forward of σ to $\mathcal{M}_{-f,g}(M)$ by Ψ. Then

(6) $$n_{\tilde{\sigma}}(x,y) = n_{\sigma}(y,x)$$

where $n_{\tilde{\sigma}(x,y)}$ (respectively $n_{\sigma}(y,x)$) is the number of points in the zero dimensional manifold $\mathcal{M}_{-f,g}(x,y)$ (respectively $\mathcal{M}_{f,g}(y,x)$) counted with respect to the orientation $\tilde{\sigma}$ (respectively σ). Let $\partial_{\tilde{\sigma}}$ and δ^{σ} be the boundary and coboundary operators defined with respect to the orientations σ and $\tilde{\sigma}$. Then

$$\begin{aligned} \langle \delta^{\sigma} \circ \Gamma(x), y \rangle &= \langle \Gamma(x), \partial_{\sigma} y \rangle \\ &= \sum_{z} n_{\sigma}(y,z) \langle \delta_x, z \rangle \\ &= n_{\sigma}(y,x) \end{aligned}$$

and

$$\begin{aligned} \langle \Gamma \circ \partial_{\tilde{\sigma}}(x), y \rangle &= \langle \delta_{\partial_{\tilde{\sigma}}(x)}, y \rangle \\ &= \sum_{z} n_{\tilde{\sigma}}(x,z) \langle \delta_z, y \rangle \\ &= n_{\tilde{\sigma}}(x,y). \end{aligned}$$

Together with (6) this proves

$$\Gamma \circ \partial_{\tilde{\sigma}} = \delta^{\sigma} \circ \Gamma.$$

Therefore, we have the isomorphism

$$\Gamma_* : H_p^{\tilde{\sigma}}(-f) \to H_{\sigma}^{n-p}(f).$$

Denote by

$$h_{\pm} : H_p^{\tilde{\sigma}}(f) \to H_p^{\tilde{\sigma}}(-f)$$

the isomorphism (3). Note that, in general, $H_p^{\sigma}(f) \neq H_p^{\tilde{\sigma}}(f)$ (see Example 4.1.4 [15] or [13]). However, for oriented M, $\partial_{\sigma} = (-1)^{p+1} \partial_{\tilde{\sigma}}$ (see Theorem 7.5 [12]) and hence the isomorphism

$$C_p(f) \to C_p(f), \ x \mapsto x$$

gives rise to the isomorphism

$$\tau_f : H_p^{\sigma}(f) \overset{\cong}{\to} H_p^{\tilde{\sigma}}(f).$$

This gives rise to *Poincaré duality isomorphism*

$$\mathrm{PD} := \Gamma_* \circ h_{\pm} \circ \tau_f : H_p^{\sigma}(f) \to H_{\sigma}^{n-p}(f).$$

3.2. **Cup product.** The classical cup product

$$\cup : H^p(M; R) \otimes H^q(M; R) \to H^{p+q}(M; R)$$

can be defined in Morse cohomology, by considering the graph moduli spaces associated to generic triple (f_1, f_2, f_3) of Morse functions on M. For a later purpose, we only sketch the main points of the construction, referring the reader to [1, 7, 8, 13] for more details. For a given triple of critical points

$$(x_1, x_2, x_3) \in \mathrm{Crit}(f_1) \times \mathrm{Crit}(f_2) \times \mathrm{Crit}(-f_3)$$

we consider the set $\mathcal{M}(f_1, f_2, f_3; x_1, x_2, x_3)$ of paths $\gamma := (\gamma_1, \gamma_2, \gamma_3)$,

$$\gamma_i : [0, \infty) \to M, \qquad i = 1, 2,$$

$$\gamma_3(-\infty, 0] \to M$$

which satisfy

$$\begin{cases} \frac{d\gamma_i}{dt} = -\nabla f_i(\gamma_i), \\ \lim_{t \to +\infty} \gamma_1(t) = x_1, \\ \lim_{t \to +\infty} \gamma_2(t) = x_2, \\ \lim_{t \to -\infty} \gamma_3(t) = x_3, \\ \gamma_1(0) = \gamma_2(0) = \gamma_3(0). \end{cases}$$

For generic choice of parameters the set $\mathcal{M}(f_1, f_2, f_3; x_1, x_2, x_3)$ is a smooth manifold of dimension $m(x_3) - m(x_2) - m(x_1)$. When $m(x_3) = m(x_1) + m(x_2)$, the zero dimensional manifold $\mathcal{M}(f_1, f_2, f_3; x_1, x_2, x_3)$ is compact. We denote the algebraic number of its points by $n(x_1, x_2, x_3)$ and define

$$\Psi(f_1, f_2, f_3) := \sum_{x_1, x_2, x_3} n(x_1, x_2, x_3) x_1 \otimes x_2 \otimes x_3 \ \in C_*(f_1) \otimes C_*(f_2) \otimes C_*(-f_3).$$

We define

$$\cup : H^p(f_1) \otimes H^q(f_2) \to H^{p+q}(f_3)$$

in the following way. For given $a_1 \in H^p(f_1)$ and $a_2 \in H^q(f_2)$ we choose their representative cycles, denoted again by a_1 and a_2. Consider the contraction cycle

$$a_1 \otimes a_2 \rfloor \Psi \quad := \quad \langle a_1 \otimes a_2, \Psi(f_1, f_2, f_3) \rangle$$
$$\in \quad C_{n-(p+q)}(-f_3)$$

and its dual cocycle

$$\Gamma(a_1 \otimes a_2 \rfloor \Psi) \ \in C^{p+q}(f_3).$$

Now $a_1 \cup a_2$ is defined to be a cohomology class of $\Gamma(a_1 \otimes a_2 \rfloor \Psi)$.

If

$$h_i^{\alpha\beta} : H^*(f_i^\alpha) \to H^*(f_i^\beta) \text{ for } i \in \{1, 2, 3\}$$

is the isomorphism dual to (3), then

(7) $h_3^{\alpha\beta}(a_1 \cup a_2) = h_1^{\alpha\beta}(a_1) \cup h_2^{\alpha\beta}(a_2).$

3.3. **Cap product.** Let M be a smooth compact n-manifold. Assume again that M is R-oriented of some ring R; Morse homology and cohomology groups will be again with coefficients in R. Let $A \subset M$ be a closed R-oriented q-dimensional submanifold and let

$$a := [A] \in H_q(M; R) \text{ and } PD(a) \in H^{n-q}(M; R)$$

be its singular homology class and its Poincaré dual. We denote

$$\mathcal{M}_{f,g}^A(x, y) := \{\gamma \in \mathcal{M}_{f,g}(x, y) \mid \gamma(0) \in A\},$$

where x, y are the critical points of f such that $\dim \mathcal{M}_{f,g}(x, y) = n - q$. By transversality, $\mathcal{M}_{f,g}^A(x, y)$ is an oriented zero dimensional manifold for generic choice of A.

By a standard compactness result in Morse theory (see for example [15]), any sequence $\gamma_k \in \mathcal{M}_{f,g}^A(x, y)$ has a subsequence converging to a family

$$(\gamma^1, \dots, \gamma^s) \in \mathcal{M}_{f,g}(x, x^1) \times \cdots \times \mathcal{M}_{f,g}(x^i, x^{i+1}) \cdots \times \mathcal{M}(x^{s-1}, y),$$

with

$$(8) \qquad m(x) < m(x^1) < \cdots < m(x^{s-1}) < m(y)$$

where m denotes the Morse index. Moreover, since $\gamma_k(0) \in A$, in a neighborhood of $0 \in \mathbb{R}$ γ_k converges to some $\gamma \in \mathcal{M}_{f,g}^A$, which means that

$$(9) \qquad \gamma^j \in \mathcal{M}_{f,g}^A(x^j, x^{j+1}) \text{ for some } j.$$

By (8) if $s > 0$, then

$$m(x^j) - m(x^{j+1}) < n$$

and hence $\dim \mathcal{M}_{f,g}^A(x^j, x^{j+1}) < 0$, which contradicts (9). Therefore $s = 0$, which means that some subsequence of γ_k converges in $\mathcal{M}_{f,g}^A(x, y)$. Hence $\mathcal{M}_{f,g}^A(x, y)$ is compact, i.e. finite.

Denote by $n_A(x, y)$ the algebraic number of points in $\mathcal{M}_{f,g}^A(x, y)$. Let

$$\psi_A : C_{n-q+r}(f) \to C_r(f)$$

be a map generated by

$$x \mapsto \sum_{y \in \mathrm{Crit}_r(f)} n_A(x, y)y, \text{ for } x \in \mathrm{Crit}_{n-q+r}(f).$$

Note that

$$\partial \circ \psi_A(x) = \partial\left(\sum_y n_A(x, y)y\right)$$

$$(10) \qquad\qquad\qquad = \sum_z \sum_y n_A(x, y)n(y, z)z$$

and

$$\psi_A \circ \partial(x) = \psi_A\left(\sum_y n(x, y)y\right)$$

$$(11) \qquad\qquad\qquad = \sum_z \sum_y n(x, y)n_A(y, z)z.$$

The numbers $n_A(x,y)n(y,z)$ and $n(x,y)n_A(y,z)$ are the algebraic numbers of points in

(12) $$\mathcal{M}^A_{f,g}(x,y) \times \mathcal{M}_{f,g}(x,z) \text{ and } \mathcal{M}_{f,g}(x,y) \times \mathcal{M}^A_{f,g}(x,z).$$

From compactness and gluing arguments (see [8, 15] for analytic details) it follows that the points in (12) are precisely the boundary points of the one dimensional manifold $\mathcal{M}^A_{f,g}(x,z)$ and hence they appear in pairs with the opposite orientation. Therefore, from (10) and (11) we deduce that ψ_A is a chain map, which gives rise to the map in homology

$$(\psi_A)_* : H_{n-q+r}(f) \to H_r(f).$$

By similar gluing and compactness arguments we conclude that, if

$$h_{\alpha\beta} : H_*(f^\alpha) \to H_*(f^\beta)$$

is the isomorphism (3), then

(13) $$h_{\alpha\beta} \circ (\psi_A)_* = (\psi_A)_* \circ h_{\alpha\beta}(x).$$

Example 5. Let us consider the special case $q = 0$. Take the generator $a \in H_0(M; R)$. The cycle representing a is a point $p \in M$, and $PD(a)$ is the volume class $\mu_M \in H^n(M; R)$. For generic choice of $p \in M$ the unique solution γ of

$$\frac{d\gamma}{dt} + \nabla^g f(\gamma) = 0, \ \gamma(0) = p$$

defines the map

$$\mathrm{Crit}_n(f) \to \mathrm{Crit}_0(f),$$

$$x \mapsto \begin{cases} \gamma(+\infty) & \text{if } x = \gamma(-\infty), \\ 0 & \text{otherwise} \end{cases}$$

which gives rise to ψ_p. In this case the map

$$(\psi_p)_* : H_n(f) \to H_0(f)$$

is an isomorphism and

$$(\psi_p)_* = \mu_M \cap \cdot$$

(see Proposition 2.6 [5] or Example 2 [1]).

3.4. **Cross product.** Another classical product in (co)homology, the cross product, is also defined in Morse category. We sketch the main points of a definition, referring the reader to [15] for details. Let M_1 and M_2 be smooth compact manifolds and

$$f_1 : M_1 \to \mathbb{R}, \ f_2 : M_2 \to \mathbb{R}$$

Morse functions. Then

$$f_1 \oplus f_2 : M_1 \times M_2 \to \mathbb{R},$$

$$f_1 \oplus f_2(x,y) := f_1(x) + f_2(y)$$

defines a Morse function on $M \times N$. The set of critical points of $f_1 \oplus f_2$ of Morse index p is

$$\mathrm{Crit}_p(f_1 \oplus f_2) = \bigcup_{p_1+p_2=p} \mathrm{Crit}_{p_1}(f_1) \times \mathrm{Crit}_{p_2}(f_2).$$

The assignment

$$\times : C_{p_1}(f_1) \otimes C_{p_2}(f_2) \to C_{p_1+p_2}(f_1 \oplus f_2)$$

defined by

$$x_1 \otimes x_2 \overset{\times}{\mapsto} (x_1, x_2), \text{ for } x_i \in Crit_{p_i}(f_i),$$

is a chain isomorphism which gives rise to the operation

$$\times : H_{p_1}(f_1) \otimes H_{p_2}(f_2) \to H_{p_1+p_2}(f_1 \oplus f_2).$$

By duality, the assignment

$$\times : C^{p_1}(f_1) \otimes C^{p_2}(f_2) \to C^{p_1+p_2}(f_1 \oplus f_2)$$

defined by

$$\langle a \times b, u \times v \rangle := \langle a, u \rangle \langle b, v \rangle$$

gives rise to the operation

$$\times : H^{p_1}(f_1) \otimes H^{p_2}(f_2) \to H^{p_1+p_2}(f_1 \oplus f_2).$$

4. VITERBO'S INVARIANTS

Let Q be a fixed quadratic function of index k on $\mathbb{R}^m$ and g_0 a standard Euclidean metric on R^m. Fix a relatively compact open set $K \subset E := M \times \mathbb{R}^m$ and Riemannian metric g_M on M and denote

$$\mathcal{S}_{(E,Q)} := \{ S : E \to \mathbb{R} \mid S(x, \xi) \equiv Q(\xi) \text{ outside } K \}$$

and

$$\mathcal{G}_{g_M \oplus g_0} := \text{ the set of metrics on } E \text{ which coincide with } g_M \oplus g_0 \text{ outside } K.$$

Morse theory applies for parameters in $\mathcal{S}_{(E,Q)} \times \mathcal{G}_{g_M \oplus g_0}$ [15]. Let $i : N \hookrightarrow M$ be a smooth closed submanifold. Denote by S_N the restriction of $S \in \mathcal{S}_{(E,Q)}$ to i^*E. Since $S = Q$ outside K, the set of critical points of S_N is finite. Furthermore, for a family $S_t^{\alpha\beta} \in \mathcal{S}_{(E,Q)}$ and for $g \in \mathcal{G}_{g_M \oplus g_0}$ the solutions of an equation

$$\frac{d\gamma}{dt} + \nabla^g (S_t^{\alpha\beta})_N(\gamma) = 0$$

which satisfy

$$(14) \qquad \int_{-\infty}^{+\infty} \left| \frac{d\gamma}{dt} \right|^2 dt < \infty$$

remain in K (note that all $S_t^{\alpha\beta}$ coincide with the same fixed Q outside K). Therefore, the construction of Morse homology groups $H_*(S_N)$ is the same as in a compact case.

For a Morse function $f : N \to \mathbb{R}$ we have obvious bijections

$$\varphi : \mathrm{Crit}(f) \overset{\cong}{\to} \mathrm{Crit}(f \oplus Q), \ x \mapsto (x, 0)$$

and

$$\mathcal{M}_{f,g_M} \cong \mathcal{M}_{f \oplus Q, g_M \oplus g_0}, \ \gamma \mapsto (\gamma, 0).$$

This gives rise to the chain isomorphism

$$\phi_p : C_p(f) \to C_{p+k}(f \oplus Q), \ \phi_p(\sum x_i) := \sum \varphi(x_i)$$

and the cochain isomorphism

$$\phi^p : C^p(f) \to C^{p+k}(f \oplus Q), \ \langle \phi_p x, \phi^p y \rangle := \langle x, y \rangle.$$

Hence, we have the isomorphisms

$$\Phi_p : H_p(f) \to H_{p+k}(f \oplus Q) \text{ and } \Phi^p : H^p(f) \to H^{p+k}(f \oplus Q).$$

Remark 6. Function $f \oplus Q$ does not belong to the space $\mathcal{S}_{(E,Q)}$. However, after composing $f \oplus Q$ with a fiber preserving diffeomorphism as in Proposition 4 we obtain the function in $\mathcal{S}_{(E,Q)}$. Since the gauge transformation does not change the Morse complex with suitably chosen metric (see the proof of Lemma 9 below for more details), we will denote the homology and cohomology groups of the new function again by $H_p(f \oplus Q)$ and $H^p(f \oplus Q)$. The same remark applies for any stabilization $S \rightsquigarrow S \oplus Q$.

By composing Φ with the connecting isomorphisms

$$h_{fS} : H_{p+k}(f \oplus Q) \to H_{p+k}(S_N)$$

and

$$h^{fS} : H^{p+k}(f \oplus Q) \to H^{p+k}(S_N)$$

we obtain the isomorphisms

$$T_p : H_p(N) \to H_{p+k}(S_N), \ T_p := h_{fS} \circ \Phi_p$$

and

$$T^p : H^p(N) \to H^{p+k}(S_N), \ T^p := h^{fS} \circ \Phi^p.$$

Note that if

$$T^\alpha_* : H_*(f) \to H_*(S^\alpha), \ T^\beta_* : H_*(f) \to H_*(S^\beta),$$

or, equivalently,

$$T^\alpha_* : H_*(f) \to H_*(g^\alpha, S), \ T^\beta_* : H_*(f) \to H_*(g^\beta, S),$$

then $h_{\alpha\beta} T^\alpha_* = T^\beta_*$.

Definition 7. Let $N \subset M$ be a closed smooth submanifold. Fix $S \in \mathcal{S}_{(E,Q)}$ such that the restriction S_N is a Morse function.

1. For $a \in H_*(N)$ we define

$$c(a, S : N) := \inf\{\lambda \mid T_* a \in \operatorname{Im}(j^\lambda_*) \subset H^\lambda_*(S_N)\}.$$

2. For $u \in H^*(N)$ we define

$$c(u, S : N) := \inf\{\lambda \mid j^*_\lambda T^* u \neq 0 \in H^*_\lambda(S_N)\}.$$

In particular, when $N = M$ we write simply $c(a, S)$ and $c(u, S)$.

Although in a definition of Morse homology (and hence of $c(\cdot, S : N)$) we use a certain Riemannian metric, it is omitted from the notation introduced in the above definition, which is justified by the following

Proposition 8. *The numbers $c(a, S : N)$ and $c(u, S : N)$ are independent of the choice of Riemannian metric $g \in \mathcal{G}_{g_M \oplus g_0}$.*

Proof. Choose $g^\alpha, g^\beta \in \mathcal{G}_{g_M \oplus g_0}$ and introduce auxiliary invariants $c(a, g^\alpha, S : N)$ and $c(u, g^\beta, S : N)$ defined via g^α and g^β. Recall that the isomorphism

$$h_{\alpha\beta} : H_*(g^\alpha, S) \to H_*(g^\beta, S)$$

is induced by the group homomorphism

$$h_\sharp : CF_*(S_N) \to CF_*(S_N),$$

(15)
$$h_\sharp(x^\alpha) = \sum_{x^\beta} n(x^\alpha, x^\beta) x^\beta,$$

where $n(x^\alpha, x^\beta)$ is the algebraic number of points in the zero-dimensional manifold $\mathcal{M}_{(g^{\alpha\beta}, S)}(x^\alpha, x^\beta)$ of solutions of equation (5) which connect x^α to x^β. We compute the difference $S(x^\beta) - S(x^\alpha)$ for every x^β which appears in sum (15), i.e for such x^β for which the set $\mathcal{M}_{(g^{\alpha\beta}, S)}(x^\alpha, x^\beta)$ is nonempty $(n(x^\alpha, x^\beta) \neq 0)$. For any $\gamma \in \mathcal{M}_{(g^{\alpha\beta}, S)}$

$$
\begin{aligned}
S(x^\beta) - S(x^\alpha) &= \int_{-\infty}^{+\infty} \frac{d}{dt} S(\gamma) dt \\
&= \int_{-\infty}^{+\infty} dS(\gamma) \frac{d\gamma}{dt} dt \\
&= \int_{-\infty}^{+\infty} \langle \nabla^{g_t^{\alpha\beta}} S, \frac{d\gamma}{dt} \rangle_{g_t^{\alpha\beta}} dt \\
&= -\int_{-\infty}^{+\infty} \left| \frac{d\gamma}{dt} \right|_{g_t^{\alpha\beta}}^2 dt \\
&\leq 0.
\end{aligned}
$$

Here we used (5). Therefore $S(x^\alpha) \geq S(x^\beta)$, which implies that $h_{\alpha\beta}$ restricts to

$$h_{\alpha\beta} : H_*^\lambda(g^\alpha, S) \to H_*^\lambda(g^\beta, S),$$

and

(16)
$$h_{\alpha\beta} \circ j_*^\lambda = j_*^\lambda \circ h_{\alpha\beta}.$$

Assume that $T_*^\alpha a \in \mathrm{Im}(j_*^\lambda)$, where

$$T_*^\alpha : H_*(N) \to H_*(g^\alpha, S).$$

Then, by (16), $h_{\alpha\beta} T_*^\alpha a \in \mathrm{Im}(j_*^\lambda)$. Since $h_{\alpha\beta} T_*^\alpha = T_*^\beta$, we have $T_*^\beta a \in \mathrm{Im}(j_*^\lambda)$ and hence

$$c(a, g^\alpha, S : N) \geq c(a, g^\beta, S : N).$$

Since the above argument is valid for any g^α, g^β, interchanging the role of g^α and g^β we get the opposite inequality and hence

$$c(a, g^\alpha, S : N) = c(a, g^\beta, S : N).$$

$\square$

In order to consider $c(a, S : N)$ as an invariant of the Lagrangian submanifold generated by S we prove that the transformations in Theorem 3 leave $c(a, \cdot : N)$ essentially unchanged. More precisely, we have the following

Lemma 9. *Let S_1 and S_2 be the two generating functions quadratic at infinity that generate the same Lagrangian submanifold $L \in T^*M$. Then there exists a constant (independent of N) $c_0 \in \mathbb{R}$ such that for any $a \in H_*(N)$ and for any $u \in H^*(N)$*

$$c(a, S_1 : N) = c(a, S_2 : N) + c_0, \ c(u, S_1 : N) = c(u, S_2 : N) + c_0.$$

Proof. By Theorem 3 and Proposition 8 it is enough to prove that with the suitable choice of metrics the Morse complex of S does not change after the stabilization $S \rightsquigarrow S \oplus Q$ and the transformation $S \rightsquigarrow S \circ \Phi + c_0$ with a fiber preserving diffeomorphism Φ and a constant $c_0 \in \mathbb{R}$. The first statement follows immediately from

$$C_p(S_N) \cong C_{p+\mathrm{index}(Q)}(S_N \oplus Q), \ \mathcal{M}_{S_N,g}(E) \cong \mathcal{M}_{S_N \oplus Q, g \oplus g_0}(E \times \mathbb{R}^l).$$

Assume now that $\Phi : E \to E$ is a fiber preserving diffeomorphism. If $S \in \mathcal{S}_{(E,Q)}$, then the transformation $S \rightsquigarrow S \circ \Phi$ does not change the Morse complex of S with the suitable choice of metric, as the following argument shows. Let g_1 be a metric in $\mathcal{G}_{g_M \oplus g_0}$ and let $g_2 := \Phi_* g_1$ be a push-forward of g_1. Then

$$\nabla^{g_1}(\Phi^* S) = \Phi^*(\nabla^{g_2} S)$$

and hence if $\gamma : \mathbb{R} \to E$ is a solution of

$$\frac{d\gamma}{dt} + \nabla^{g_1} S_N(\gamma) = 0,$$

then $\tilde{\gamma} := \Phi \circ \gamma$ satisfies

$$
\begin{aligned}
\frac{d\tilde{\gamma}}{dt} + \nabla^{g_2} S_N(\tilde{\gamma}) &= T\Phi\left(\frac{d\gamma}{dt} + (T\Phi)^{-1}\nabla^{g_2} S_N(\tilde{\gamma})\right) \\
&= T\Phi\left(\frac{d\gamma}{dt} + \nabla^{g_1} S_N(\gamma)\right) \\
&= 0.
\end{aligned}
$$

Therefore

$$\mathcal{M}_{S_N, g_1}(E) \cong \mathcal{M}_{\Phi^* S_N, g_2}(E).$$

In particular, there exists a compact set K_0 such that all gradient trajectories (in metric $\Phi_* g_1$) which satisfy (14) remain in K_0. Note that for every negative gradient trajectory γ starting at the point $x \in \mathrm{Crit}(S \circ \Phi)$ and leaving K_0 (i.e. not satisfying (14))

$$\lim_{\tau \to \infty} S \circ \Phi(\gamma(\tau)) = -\infty.$$

Hence there exists a relatively compact open set $U_1 \supset K_0$ such that for any such γ and any $\tau \in \mathbb{R}$ such that $\gamma(\tau) \notin U_1$

(17) $\qquad S \circ \Phi(\gamma(\tau)) > \min\{S \circ \Phi(x) \mid x \in \mathrm{Crit}(S \circ \Phi)\}.$

Choose a relatively compact open set $U_2 \supset \overline{U}_1$ and define the metric g_3 by

$$g_3 = \begin{cases} \Phi_* g_1 & \text{in } U_1, \\ g_1 & \text{outside } \overline{U}_2. \end{cases}$$

Recall that all gradient trajectories which satisfy (14) end at the critical point of $S \circ \Phi$. Since $S \circ \Phi$ decreases along negative gradient trajectories, it follows from (17) and the construction of a metric g_3 that all g_3-gradient trajectories of $S \circ \Phi$ satisfying (14) remain in U_1. Therefore

$$\mathcal{M}_{S_N, g_1}(E) \cong \mathcal{M}_{\Phi^* S_N, g_3}(E)$$

and

$$C_p(S_N) \cong C_p(S_N \circ \Phi).$$

Note that, since $\nabla S = \nabla(S + c_0)$ for a constant $c_0 \in \mathbb{R}$, Morse theory also applies to the functions in

$$\mathcal{S}_{(E,Q)} + c_0 := \{S + c_0 \mid S \in \mathcal{S}_{(E,Q)}\}.$$

Hence $c(a, S + c_0 : N)$ is well defined and, obviously,

$$c(a, S + c_0 : N) = c(a, S : N) + c_0.$$

This finishes the proof. $\square$

The following theorem establishes the main properties of c:

Theorem 10 (Compare with [20]). *Let $N \subset M$ and S_N be as before. Denote the conormal bundle of N in T^*M by ν^*N. Let $L_S \subset T^*M$ be a Lagrangian submanifold generated by S.*

1. *$c(a, S : N)$ and $c(u, S : N)$ are the critical values of S_N.*
2. *For $a, b \in H_*(N)$ there exist points $x_a, x_b \in \nu^*N \cap L_S$ such that*

$$c(a, S : N) - c(b, S : N) = \int_\gamma \theta$$

    for any smooth path γ in L_S connecting x_a to x_b.
3. *The function $c(a, \cdot : N) : S \mapsto c(a, S : N)$ from $\mathcal{S}_{(E,Q)}$ to $\mathbb{R}$ is C^0-continuous. Hence, we can define $c(a, S : N)$ for any (not necessarily generic) S by*

$$c(a, S : N) := \lim_{S_n \to S} c(a, S_n : N).$$

4. *If $S_i : E_i \to \mathbb{R}$, $i = 1, 2$ are two generating functions quadratic at infinity and*

$$S^3 := S^1 \sharp S^2 : E_1 \oplus E_2 \to \mathbb{R},$$

$$S^3(\zeta_1 \oplus \zeta_2) = S^1(\zeta_1) + S^2(\zeta_2),$$

    then

$$c(u \cup v, S^3 : N) \geq c(u, S^1 : N) + c(v, S^2 : N)$$

    for any $u, v \in H^*(N)$.
5. *If N is oriented, then*

$$c(\Gamma_*(a), S : N) = -c(a, -S : N)$$

    where Γ_* is as in Section 3.1.
6. *If M is oriented and connected and $\mu \in H_n(M)$ and $1 \in H_0(M)$ are the generators of $H_n(M)$ and $H_0(M)$, then*

$$c(\mu, S : M) \geq c(1, S : M)$$

    and $c(\mu, S : M) = c(1, S : M)$ if and only if S generates the zero section o_M.

Statements 5. and 6. in non-orientable cases hold in $\mathbb{Z}_2$ coefficients.

Proof. We give a proof of this theorem using the machinery of Morse homology, motivated by the ideas of [14, 13]. For the proof of analogous theorem by methods of algebraic and differential topology we refer the reader to the original paper [20].

Since $S - Q$ has compact support, there are only finitely many critical points of S. In particular,

$$H_*^{-\lambda}(S_N) = 0 \text{ and } H_*^{\lambda}(S_N) = H_*^{\infty}(S_N)$$

for λ large enough. Hence $c \neq \pm\infty$ and thus 1. follows from the definition and the fact that

$$j_* : H_*^{\lambda}(S_N) \to H_*^{\lambda+\varepsilon}(S_N)$$

is an isomorphism if there are no critical values of S in interval $[\lambda, \lambda + \varepsilon]$. Since $i_S(\mathrm{Crit}(S_N)) = \nu^* N \cap L_S$, 2. is a consequence of 1. and the identity $i_S^*\theta = d(S|_{\Sigma_S})$.

Now we prove 3. For Morse functions $S^{\alpha}, S^{\beta} \in \mathcal{S}_{(E,Q)}$ consider the connecting homotopy

$$S^{\alpha\beta} = \rho S^{\beta} + (1 - \rho)S^{\alpha} \in \mathcal{S}_{(E,Q)}$$

where

$$\rho : \mathbb{R} \to \mathbb{R}$$

is a C^{∞} function such that

$$\rho(\tau) = \begin{cases} 1 & \text{for } \tau \geq 1, \\ 0 & \text{for } \tau \leq 0. \end{cases}$$

Then, we compute $S^{\beta}(x^{\beta}) - S^{\alpha}(x^{\alpha})$ for a pair

$$(x^{\alpha}, x^{\beta}) \in \mathrm{Crit}(S^{\alpha}) \times \mathrm{Crit}(S^{\beta})$$

connected by the trajectory $\gamma \in \mathcal{M}_{S^{\alpha\beta},g}(x^{\alpha}, x^{\beta})$. We obtain

$$
\begin{aligned}
S^{\beta}(x^{\beta}) - S^{\alpha}(x^{\alpha}) &= \int_{-\infty}^{+\infty} \frac{d}{d\tau} S^{\alpha\beta}(\gamma(\tau))d\tau \\
&= \int_{-\infty}^{+\infty} [dS^{\alpha\beta}(\gamma(\tau))\frac{d\gamma}{d\tau} + \rho'(\tau)(S^{\beta}(\gamma(\tau)) - S^{\alpha}(\gamma(\tau)))]d\tau \\
&\leq -\int_{-\infty}^{+\infty} \langle \nabla S^{\alpha\beta}(\gamma(\tau)), \frac{d\gamma}{d\tau}\rangle d\tau + \max(S^{\beta} - S^{\alpha}) \\
&= -\int_{-\infty}^{+\infty} |\nabla S^{\alpha\beta}(\gamma(\tau))|^2 d\tau + \max(S^{\beta} - S^{\alpha}) \text{ (by 4)} \\
&\leq \max(S^{\beta} - S^{\alpha}).
\end{aligned}
$$

The same estimate holds for any regular connecting homotopy sufficiently close to $S^{\alpha\beta}$. Hence, we have the commutative diagram

$$
\begin{array}{ccc}
H_*^{\lambda}(S^{\alpha}) & \xrightarrow{j_*^{\lambda}} & H_*(S^{\alpha}) \\
\downarrow{h_{\alpha\beta}} & & \downarrow{h_{\alpha\beta}} \\
H_*^{\lambda_{\alpha\beta}}(S^{\beta}) & \xrightarrow{j_*^{\lambda_{\alpha\beta}}} & H_*(S^{\beta})
\end{array}
$$

where $\lambda_{\alpha\beta} := \lambda + \max(S^{\beta} - S^{\alpha})$. Therefore, using the definition of c we deduce

$$(18) \qquad c(a, S^{\beta} : N) - c(a, S^{\alpha} : N) \leq \max(S^{\beta} - S^{\alpha}).$$

Interchanging the role of α and β we get

$$c(a, S^\alpha : N) - C(a, S^\beta : N) \leq \max(S^\alpha - S^\beta) = -\min(S^\beta - S^\alpha)$$

and hence

(19)
$$c(a, S^\beta : N) - c(a, S^\alpha : N) \geq \min(S^\beta - S^\alpha).$$

Combining (18) and (19) we get

$$|c(a, S^\beta : N) - c(a, S^\alpha : N)| \leq \|S^\beta - S^\alpha\|_{C^0}.$$

To prove 4., we first assume that S^1, S^2, S^3 are (after stabilization) defined on $M \times \mathbb{R}^{m_i}$ for $i \in \{1, 2, 3\}$. Further, we extend S^1 and S^2 to whole $E := M \times \mathbb{R}^{m_1 + m_2}$ by

$$S^1(q, \xi, \eta) = S^1(q, \xi) + \varepsilon|\eta|^2, \ \ S^2(q, \xi, \eta) = S^2(q, \eta) + \varepsilon|\xi|^2,$$

where $|\zeta|^2 = \sum \zeta_j^2$ is the standard norm on $\mathbb{R}^m$ and $\varepsilon > 0$. Since the stabilization $S \rightsquigarrow S \oplus Q$ does not change the value of $c(\cdot, S : N)$, we will prove 4. for $S^1(q, \xi, \eta)$, $S^2(q, \xi, \eta)$ and

(20)
$$S^3(q, \xi, \eta) = S^1(q, \xi) + S^2(q, \eta) \left(= S^1(q, \xi, \eta) + S^2(q, \xi, \eta) - \varepsilon(|\xi|^2 + |\eta|^2)\right).$$

Note that, if

$$T^{p_1} : H^{p_1}(N) \to H^{p_1 + k_1}(S_N^1), \ T^{p_2} : H^{p_2}(N) \to H^{p_2 + k_2}(S_N^2)$$

and

$$T^p : H^p(N) \to H^{p+k}(S_N^3), \ p = p_1 + p_2, \ k = k_1 + k_2,$$

then

(21)
$$T^p(u \cup v) = T^{p_1}(u) \cup T^{p_2}(v).$$

Indeed, for $S^i = f \oplus Q^i$ $(i \in \{1, 2\})$, where f is a Morse function on N, this equality is obvious since the stabilization $f \rightsquigarrow f \oplus Q$ does not change the Morse chain complex of f. Furthermore, if

$$h^{\alpha\beta} : H^{p+k}(f \oplus Q^i) \to H^{p+k}(S_N^i)$$

is the canonical isomorphism, then, by (7)

$$h^{\alpha\beta}(T^{p_1}(u) \cup T^{p_2}(v)) = h^{\alpha\beta}(T^{p_1}(u)) \cup h^{\alpha\beta}(T^{p_2}(v))$$

and thus (21) follows.

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be the solution of

$$\begin{cases} \frac{d\gamma_i}{dt} = -\nabla S^i(\gamma_i), \\ \lim_{t \to +\infty} \gamma_1(t) = x_1, \\ \lim_{t \to +\infty} \gamma_2(t) = x_2, \\ \lim_{t \to -\infty} \gamma_3(t) = x_3, \\ \gamma_1(0) = \gamma_2(0) = \gamma_3(0). \end{cases}$$

Note that, since all S_i are quadratic outside fixed compact set, we have $\gamma_i \subset K_0$ for some fixed compact set K_0. If we write $\gamma_1, \gamma_2 \subset M \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ as $\gamma_i(t) = (x_i(t), \xi_i(t), \eta_i(t))$ for $i \in \{1, 2\}$, then

$$
\begin{aligned}
S^1(x_1) + S^2(x_2) - S^3(x_3) &= S^1(x_1) - S^1(\gamma(0)) + S^2(x_2) - S^2(\gamma(0)) \\
&\quad - S^3(x_3) + S^3(\gamma(0)) + \varepsilon(|\xi_1(0)|^2 + |\eta_2(0)|^2) \\
&\leq \varepsilon \max_{K_0}(|\xi|^2 + |\eta|^2) \\
&=: \varepsilon_1.
\end{aligned}
$$

Here we used $S^1(\gamma_1(0)) + S^2(\gamma_2(0)) = S^3(\gamma_3(0)) + \varepsilon(|\xi_1(0)|^2 + |\eta_2(0)|^2)$ (by (20), since $\gamma_1(0) = \gamma_2(0) = \gamma_3(0)$) and the fact that S^i decreases along γ_i. Therefore,

$$-S^3(x_3) \leq -S^1(x_1) - S^2(x_2) + \varepsilon_1.$$

Assume that

(22) $$c(u, S^1 : N) + c(v, S^2 : N) > \lambda$$

and choose λ_1, λ_2 such that $\lambda < \lambda_1 + \lambda_2$ and $\lambda_1 < c(u, S^1 : N)$, $\lambda_2 < c(v, S^2 : N)$. Choose the cocycles representing $U = T^*u \in H^*(S_N^1)$ and $V = T^*v \in H^*(S_N^2)$, denoted again by U and V. We have

$$U(x) = 0, \; V(y) = 0, \text{ for } x \in \mathrm{Crit}^{\lambda_1}(S_N^1), \; y \in \mathrm{Crit}^{\lambda_2}(S_N^2).$$

Hence, the only nontrivial contributions to

$$U \otimes V \rfloor \Psi(S_N^1, S_N^2, S_N^3) := \sum_{x_1, x_2, x_3} n(x_1, x_2, x_3)\langle U, x_1 \rangle \langle V, x_2 \rangle x_3$$

come from the triples (x_1, x_2, x_3) with

$$-S^3(x_3) \leq -S^1(x_1) - S^2(x_2) + \varepsilon_1 < -\lambda_1 - \lambda_2 + \varepsilon_1.$$

Therefore

$$U \otimes V \rfloor \Psi(S_N^1, S_N^2, S_N^3) \in C_*^{-\lambda_1 - \lambda_2 + \varepsilon_1}(-S_N^3).$$

By definition of Γ (see Section 3.1), we have

$$\Gamma(U \otimes V \rfloor \Psi)\big|_{C_*^{\lambda_1 + \lambda_2 - \varepsilon_1}(S_N^3)} \equiv 0.$$

Therefore $j_{\lambda - \varepsilon_1}^*(T^*u \cup T^*v) = 0$, i.e.

$$\lambda - \varepsilon_1 \leq c(u \cup v, S^3 : N)$$

for every λ satisfying (22). Hence

$$c(u \cup v, S^3 : N) + \varepsilon_1 \geq c(u, S^1 : N) + c(v, S^2 : N).$$

Letting $\varepsilon \to 0$ (and hence $\varepsilon_1 \to 0$) we prove 4.

For the proof of 5. we consider the groups

$$C_*^{(\lambda, \mu)}(S) := C_*^{\mu}(S)/C_*^{\lambda}(S)$$

and

$$C_{(\lambda, \mu)}^*(S) := \mathrm{Hom}(C_*^{(\lambda, \mu)}(S), R).$$

The boundary operator ∂ induces

$$\partial_p^{(\lambda, \mu)} : C_p^{(\lambda, \mu)}(S) \to C_{p-1}^{(\lambda, \mu)}(S)$$

and

$$\delta^p_{(\lambda,\mu)} := \operatorname{Hom}(\delta_p^{(\lambda,\mu)}) : C^p_{(\lambda,\mu)}(S) \to C^{p+1}_{(\lambda,\mu)}(S).$$

Hence we can define the relative homology and cohomology groups

$$H_p^{(\lambda,\mu)}(S) := \operatorname{Ker}(\partial_p^{(\lambda,\mu)})/\operatorname{Im}(\partial_p^{(\lambda,\mu)}),$$

$$H^p_{(\lambda,\mu)}(S) := \operatorname{Ker}(\delta^p_{(\lambda,\mu)})/\operatorname{Im}(\delta^{p-1}_{(\lambda,\mu)}).$$

Note that, by definition of Γ_*,

$$\Gamma_* : H_*^{(-\mu,-\lambda)}(S) \to H^*_{(\lambda,\mu)}(-S)$$

with homology and cohomology groups defined with respect to the orientations σ and $\tilde{\sigma}$ as in Section 3.1. The short exact sequences

$$0 \to C_*^{(-\infty,-\lambda)}(S) \to C_*(S) \to C_*^{(-\lambda,+\infty)}(S) \to 0$$

and

$$0 \to C^*_{(\lambda,+\infty)}(S) \to C^*(S) \to C^*_{(-\infty,\lambda)}(S) \to 0$$

generated by obvious inclusion maps give rise to the long exact sequences in homology and cohomology such that the diagram

$$
\begin{array}{ccc}
H_*^{(-\infty,-\lambda)}(S) & \xrightarrow{\Gamma_*} & H^*_{(\lambda,\infty)}(-S) \\
\downarrow & & \downarrow \\
H_*(S) & \xrightarrow{\Gamma_*} & H^*(-S) \\
\downarrow & & \downarrow \\
H_*^{(-\lambda,\infty)}(S) & \xrightarrow{\Gamma_*} & H^*_{(-\infty,\lambda)}(-S)
\end{array}
$$

commutes. By exactness of vertical columns, $j_\lambda^* \Gamma_*(T_* a) \neq 0$ is equivalent to $T_* a \in \operatorname{Im}(j_*^{-\lambda})$. Therefore, if $\lambda < c(\Gamma_*(a), -S : N)$, then $-\lambda \geq c(a, S : N)$ and thus $\lambda \leq -c(a, S : N)$. This proves

$$c(\Gamma_*(a), -S : N) \leq -c(a, S : N).$$

Conversely, if $\lambda < -c(a, S : N)$, then, by definition of c, $T^* a \in \operatorname{Im}(j_*^{-\lambda})$. By exactness $j_\lambda^* \Gamma_*(T_*(a)) = 0$ and thus $\lambda \leq c(\Gamma_*(a), S : N)$. That proves

$$-c(a, S : N) \leq c(\Gamma_*(a), S : N).$$

This finishes the proof of 5.

In order to prove 6. we first extend the definition of cap product given in Section 3. Consider a point $z \in M$ and the fiber $\pi^{-1}(z) \subset E$. For $x \in C_{n+k}(S)$ and $y \in C_k(S)$ we define

$$\mathcal{M}^z_{S,g}(x,y) := \{\gamma \in \mathcal{M}_{S,g}(x,y) \mid \gamma(0) \in \pi^{-1}(z)\}.$$

Consider the evaluation map

$$Ev : \mathcal{M}_{S,g}(x,y) \to M,$$

$$Ev(\gamma) = \pi(\gamma(0)),$$

For a regular value z of Ev $\mathcal{M}^z_{S,g}(x,y)$ is a smooth zero dimensional manifold. Since all bounded gradient trajectories defined by parameters from $\mathcal{S}_{(E,Q)} \times \mathcal{G}_{g_M \oplus g_0}$ remain in a fixed compact subset of E, the compactness of $\mathcal{M}^z_{S,g}(x,y)$ follows from

the same arguments as in Section 3. Hence we can define $n_z(x, y)$ as the algebraic number of points in $\mathcal{M}^z_{S,g}(x, y)$ and the map

$$\psi_z : C_{n+k}(S) \to C_k(S)$$

as the linear extension of the map

$$x \mapsto \sum_\beta n_z(x, y) y.$$

The chain property of ψ_z follows as in Section 3 and thus ψ_z descends to the map in a homology, denoted again by

$$\mu_M \cap \cdot : H_{n+k}(S) \to H_k(S).$$

Moreover, the following diagram commutes:

$$
\begin{array}{ccc}
H_{n+k}(S) & \xrightarrow{\mu_M \cap \cdot} & H_k(S) \\
\uparrow h_{\alpha\beta} & & \uparrow h_{\alpha\beta} \\
H_{n+k}(f \oplus Q) & \xrightarrow{\mu_M \cap \cdot} & H_k(f \oplus Q) \\
\uparrow & & \uparrow \\
H_n(f) & \xrightarrow{\mu_M \cap \cdot} & H_0(f)
\end{array}
$$

and all the arrows are the isomorphisms. For the lower rectangle this is again the consequence of the fact that the stabilization $f \rightsquigarrow f \oplus Q$ does not change the Morse chain complex of f; for the upper one this is the consequence of (13).

If S does not generate o_M, then there exists a point $z_0 \in M$ such that $dS(e) \neq 0$ for all $e \in \pi^{-1}(z_0)$. Since $S = Q$ outside a compact set, that means that

$$\inf_{e \in \pi^{-1}(z_0)} |\nabla S| = 2\varepsilon_0 > 0.$$

Hence, there exist an ε_1-ball $U \subset M$ with the center at z_0 and a C^1 neighborhood $\mathcal{U} \subset \mathcal{S}_{(E,Q)}$ of S such that

$$(23) \qquad\qquad \inf_{e \in \pi^{-1}(U)} |\nabla S_0| > \varepsilon_0 > 0$$

for any $S_0 \in \mathcal{U}$. Let $S_l \in \mathcal{U}$ be a sequence of generic functions such that

$$(24) \qquad\qquad C^1 - \lim_{l \to \infty} S_l = S.$$

Choose a generic $z_l \in U(:= B_{\varepsilon_1}(z_0))$ which is $\frac{\varepsilon_1}{2}$-close to z_0. Let γ_l be a negative gradient trajectory of S_l with $\gamma_l(0) \in \pi^{-1}(z_l)$. By (23) $|\nabla S_l(\gamma_l(0))| > \varepsilon_0 > 0$. Since

$$\lim_{\tau \to \pm\infty} \nabla S_l(\gamma_l(\tau)) = 0,$$

there exist $\tau_l \in \mathbb{R}$ such that

$$\gamma_l(\tau_l) \notin \pi^{-1}(U),$$

and thus

$$
\begin{aligned}
S_l(\gamma_l(+\infty)) - S_l(\gamma_l(-\infty)) &= \int_{-\infty}^{+\infty} \frac{d}{d\tau} S_l(\gamma_l(\tau)) d\tau \\
&= -\int_{-\infty}^{+\infty} |\nabla S_l(\gamma_l(\tau))|^2 \, d\tau \\
&\leq -\int_0^{+\tau_l} \left| \frac{d\gamma_l}{d\tau} \right|^2 d\tau \\
&\leq -(\mathrm{dist}(z_l, M \setminus U))^2 \\
&\leq -\frac{\varepsilon_1^2}{4} =: -\varepsilon.
\end{aligned}
$$

Hence from the non-degeneracy of cap action we obtain the isomorphism

$$
\mu_M \cap \cdot : H_{n+k}^\lambda(S_l) \to H_k^{\lambda-\varepsilon}(S_l).
$$

Since $H_0(M)$ and $H_n(M)$ are generated by one generator, from the commutative diagram

$$
\begin{array}{ccc}
H_{n+k}^\lambda(S_l) & \xrightarrow{j_*^\lambda} & H_{n+k}(S_l) \\
\downarrow{\mu_M \cap \cdot} & & \downarrow{\mu_M \cap \cdot} \\
H_k^{\lambda-\varepsilon}(S_l) & \xrightarrow{j_*^{\lambda-\varepsilon}} & H_k(S_l)
\end{array}
$$

we conclude $c(\mu, S_l : M) - c(1, S_l : M) > \varepsilon$. Now 6. follows by (24) and the continuity of $c(\mu, \cdot : M) - c(1, \cdot : M)$. $\qquad\square$

Acknowledgments

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